Conditioning, Conditional Independence and Irrelevance in Evidence Theory

Jiřina Vejnarová

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic

ISIPTA'11, Innsbruck

Background

- Institute of Information Theory and Automation of the Academy of Sciences of the Czech Republic (originally Czechoslovak Academy of Sciences) since the end of 1986

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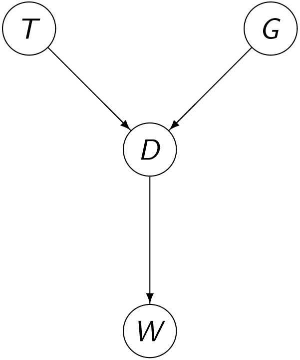
- ISIPTA'07, ISIPTA'09, ISIPTA'11 — multidimensional models in evidence theory

...

Motivation



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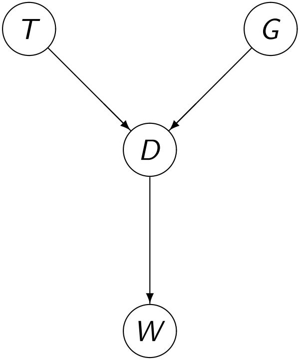

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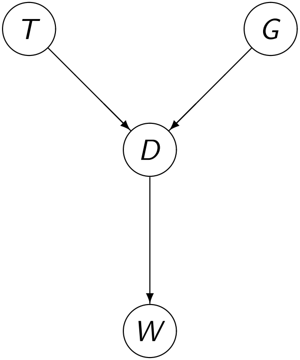

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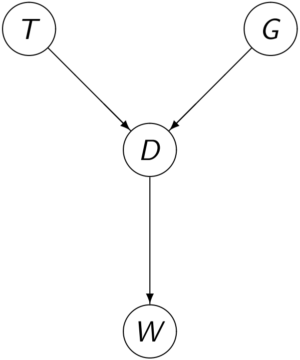

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Fact

In (precise) probabilistic setting conditional independence of X and Y given Z:

$$P(XYZ) \cdot P(Z) = P(XZ) \cdot P(YZ)$$

is (more or less) equivalent to conditional irrelevance:

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BUT IS IT TRUE ALSO IN EVIDENCE THEORY???

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- *Formal properties:* to decompose the multidimensional models into marginals or factors, to simplify inference.
- *Framework preservation:* to obtain multidimensional model in the same framework as the marginals.
- *Consistency with marginalization:* to obtain multidimensional models keeping their marginals.

Independence

Let m be a basic assignment on $\mathbf{X}_N$ and $K, L \subset N$ be disjoint. We say that groups of variables X_K and X_L are *independent with respect to basic assignment m* (and denote it by $K \perp\!\!\!\perp L\ [m]$) if

$$m^{\downarrow K \cup L}(A) = m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L})$$

for all $A \subseteq \mathbf{X}_{K \cup L}$ for which $A = A^{\downarrow K} \times A^{\downarrow L}$, and $m(A) = 0$ otherwise.

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random set independence (Couso, Moral Walley)

non-interactivity (Klir at al.)

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Lemma

Let K, L be disjoint, then $K \perp\!\!\!\perp L\ [m]$ if and only if

$$Q^{\downarrow K \cup L}(A) = Q^{\downarrow K}(A^{\downarrow K}) \cdot Q^{\downarrow L}(A^{\downarrow L})$$

for all $A \subseteq \mathbf{X}_{K \cup L}$.

Conditional non-interactivity

Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. Groups of variables X_K and X_L are *conditionally non-interactive given X_M with respect to m* (Ben Yaghlane et al., IJAR 2002) (and denote it by $K \perp\!\!\!\perp L|M\ [Q]$) if and only if the equality

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conditional independence (Shenoy, IJAR 1994; Studený, ECSQARU'93)

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A *join* of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ is the set

$$A \bowtie B = \{x \in \mathbf{X}_{K\cup L} : x^{\downarrow K} \in A \ \ \&\ \ x^{\downarrow L} \in B\}.$$

Irrelevance

Group of variables X_L is *irrelevant* to X_K $(K \cap L = \emptyset)$ if for any $B \subseteq \mathbf{X}_L$ such that $Pl(B) > 0$ (or $Bel(B) > 0$)

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- Dempster's conditioning rule
- Focusing
- Many other conditioning rules (e.g. by Fagin and Halpern, UAI'91)

Conditioning rules

Dempster's rule of conditioning

$$m(A|B) = \frac{\displaystyle\sum_{C \subseteq \mathbf{X}_N : C \cap B = A} m(C)}{Pl(B)}$$

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Focusing

$$m(A||B) = \begin{cases} \dfrac{m(A)}{Bel(B)} & \text{if} \quad A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

$B \subseteq \mathbf{X}_N$ such that $Bel(B) > 0$.

Conditioning rules

Dempster's rule of conditioning

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\begin{aligned}
Bel(A|B) &= \frac{Bel(A \cup B^C) - Bel(B^C)}{1 - Bel(B^C)}, \\
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- None of them is symmetric, in general.
- Even in case of symmetry none of them implies independence.

Conditional irrelevance

Group of variables X_L is *conditionally irrelevant* to X_K given X_M (K, L, M disjoint, $K \neq \emptyset \neq L$) if for any $B \subseteq \mathbf{X}_{L \cup M}$ such that $Pl(B) > 0$ ($Bel(B) > 0$, respectively)

$$m_{X_K|X_L X_M}(A|B) = m_{X_K|X_M}(A|B^{\downarrow M}).$$

Properties (unconditional case)

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Theorem

Let X_K and X_L be conditionally independent groups of variables given X_M under joint basic assignment m on $\mathbf{X}_{K\cup L\cup M}$ $(K, L, M$ disjoint, $K \neq \emptyset \neq L)$. Then

$$m_{X_K||X_LX_M}(A||B) = m_{X_K||X_M}(A||B^{\downarrow M})$$

for any $m^{\downarrow L\cup M}$-atom $B \subseteq \mathbf{X}_{L\cup M}$ such that $B^{\downarrow M}$ is $m^{\downarrow M}$-atom and $A \subseteq \mathbf{X}_K$.

Instead of conclusions...

... just an idea

Let X_K and X_L ($K \cap L = \emptyset$) be two groups of variables with values in $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively. Then the *conditional basic assignment* of X_K given $X_L \in B \subseteq \mathbf{X}_L$ (for B such that $m(B) > 0$) is defined as follows:

$$m_{X_K|X_L}(A|B) \;=\; \frac{\displaystyle\sum_{C \subseteq \mathbf{X}_{K\cup L}: C^{\downarrow L}=B \,\&\, C^{\downarrow K}=A} m(C)}{m(B)}$$

for any A.

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Although an analogical conditioning rule for events has not a sense (see Example 1 in the Proceedings), just suggested conditioning rule seems to have the following properties:

- $m_{X_K|X_L}(A|B) \in [0,1]$ for any $A \subseteq \mathbf{X}_K$ and any $B \subseteq \mathbf{X}_L$ such that $m(B) > 0$.
- For a fixed $B \subseteq \mathbf{X}_L$ such that $m(B) > 0$

$$\sum_{A \subseteq \mathbf{X}_K} m_{X_K|X_L}(A|B) = 1.$$